The Filter Wizard
issue 36: AC Voltage gain using just resistors and capacitors?
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You'll know that achieving AC voltage gain in a circuit is a trivial matter, when you are permitted to use a power supply and some active components, such as op-amps, transistors, or even – yes, there is still a thriving user community for these – vacuum tubes (valves, if you're a Brit). But what approach can we take if no power supply is allowed, or we need this to work at 1000 degrees. Most vexing of all: imagine that this is an interview question for that plum analogue job you've been after for so long!

Well, it's also straightforward to get voltage gain from wound components exploiting magnetic induction. We're all familiar with using a transformer to step down the AC line voltage in a power supply, prior to rectification and regulation. Voltage increase is just as easy; less commonly used in today's subminiaturized world, the humble transformer was once the solution of choice for increasing the magnitude of the small audio signal from a moving coil microphone or a phono cartridge. There's no power gain in this passive arrangement, of course; the higher voltage is available from a correspondingly higher source impedance, and there's no increase in the maximum power that can be extracted from the source.

At high enough frequencies, the very propagation of EM waves over carefully shaped conductors and insulators can have the same effect. Baluns made out of such striplines are used widely in radio frequency engineering to match power delivery into unequal impedances. "I knew that!", all you engineers out there are saying to yourselves.

But what if you are not permitted to use magnetic induction at all, so no transmission lines or even inductors, but we (or your evil interrogator, sitting across the desk from you, languidly clicking his ballpoint pen…) still want AC voltage gain between two ports in your circuit?

Let's say that you have a supply of resistors and capacitors, but nothing else. Also, of course, that these components and the board you'll be using are dimensionally minuscule compared to the free-space wavelength of the nominal frequency you'll be working at, just to make sure that you don't exploit any electromagnetic phenomena.

Here's how to pose the question in Filter terms: Can you make a network of resistors and capacitors having a port-to-port frequency response magnitude, i.e. gain, that's greater than unity, over some finite frequency range. For extra credit, can you make that circuit have a gain of greater than unity at ALL finite frequencies?

Stuck yet? It's mean of me, I know. I want to get all of that out of my system before Christmas. And, speaking of Christmas – rejoice! Indeed this is possible, provided that we don't get too greedy about just how high a gain value we want. At this point, stop and think about it for a little, before cheating and reading the rest of the column…

OK, let's go through the behaviour of some simple RC circuits, step by step. First stop is figure 1. It's just an RC ladder with two equal resistors of unit value, and two equal capacitors of unit value. We'll see that the resulting transfer function with these component values is a simple second order lowpass response with a Q factor of 0.3333 and a pole frequency of 1/(2pi) Hz. Figure 2 shows a gain that's clearly less than unity at all frequencies displayed; it tends to unity gain (zero dB, in other words) at DC.

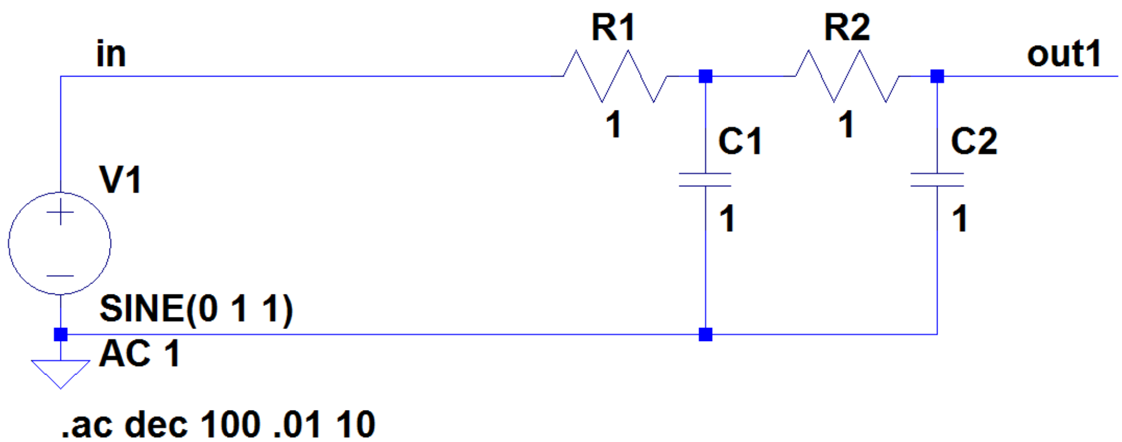

Figure 1: A simple RCRC lowpass ladder with unit components.

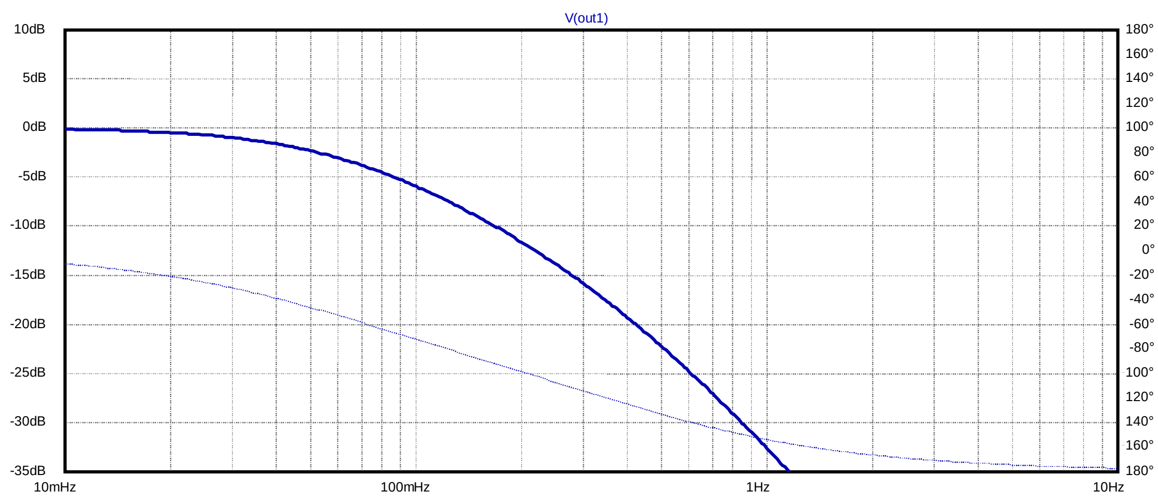

Figure 2: frequency response of the circuit in figure 1.

Incidentally, second-order filter sections with Q factors of 0.5 or less can be a bit confusing for some people. The second-order polynomial in **s** that forms the denominator of the transfer function can be factorized into two real roots (they are both negative). So, such a transfer function could be expressed as the product of two first-order sections, and implemented by a cascade of two first order RC sections, either buffered from each other or just implemented as an RC ladder like this. But this is still a second order transfer function, and it still has an equivalent Q factor, just one that is less than 0.5.

Two conditions must be satisfied for the Q factor of the network to tend to its highest possible value of 0.5. Firstly, the RC products of the first and second ‘halves’ of the network must be the same. Secondly, the impedance level of the second half must get very high compared to that of the first half. You can see this when you derive the transfer function. I cheated this time and used SapWin to do this, and the computed transfer function is shown in the screenshot of figure 3.

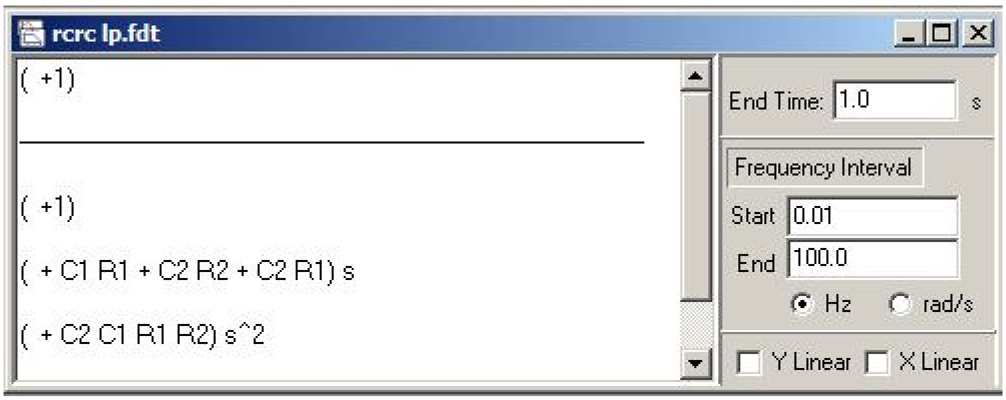

Figure 3: A screenshot from SapWin, showing the symbolic analysis of figure 1’s circuit.

I think I’ll leave it as a homework assignment (I know I said I would stop doing that, but once in a while won’t hurt ya) to prove that the maximum Q value you can get from the expression shown in figure 3 is 0.5, whatever values of resistor and capacitor you choose. But it’s pretty easy to see that if both Rs and both Cs are unity, that **s** term in the denominator has a value of 3, meaning a Q factor of 0.3333.

So far, so dull, you’ll be thinking. No possible combination of Rs and Cs is ever going to make this circuit do anything interesting. Hey, look over there! <snips wires and swaps two connections while audience is distracted>. I just did something to the circuit and it definitely has a little peak in its response now – check out figure 4:

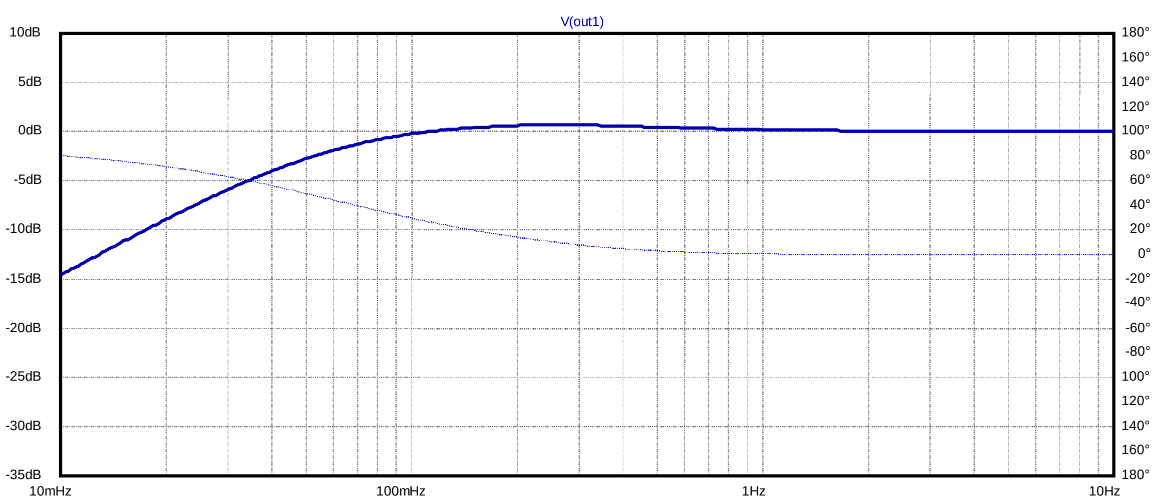

Figure 4: Same components as used in figure 1, just swapped two connections.

If we were to zoom in, we'd see that the network gain peaks at +0.65 dB. Not much, but it's definitely gain! And I didn't add, remove nor change any components neither, gov'nor.

If the suspense is unbearable, look at figure 5 to see what I did. I just flipped the connections to the ladder, taking the output voltage between the input terminal of the ladder and its end terminal, rather than between the end terminal and ground.

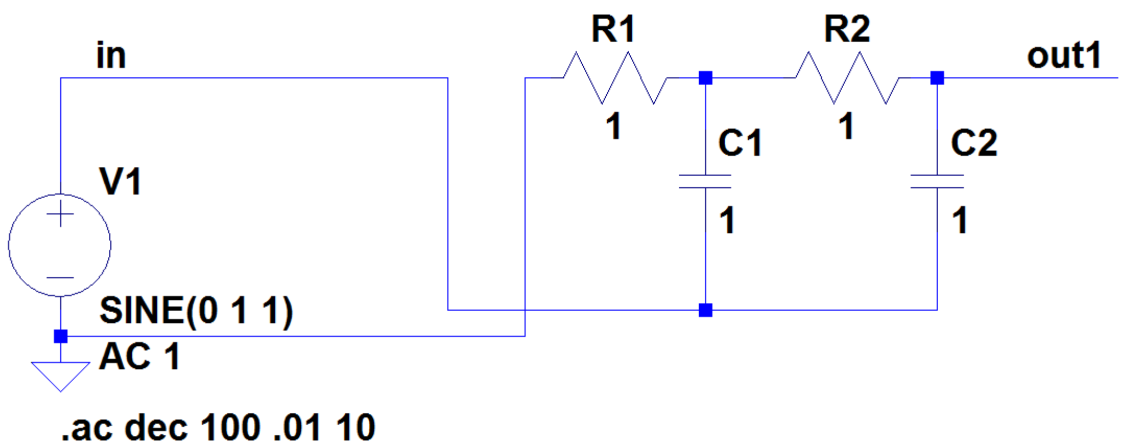

Figure 5: The circuit that gave the response in figure 4; compare figure 1 to see the swap.

The overall response now has a highpass character; the difference between input and output voltage falls to zero as the frequency tends to DC. Very high frequencies pass through unattenuated. The transfer function of this modified connection has the same poles as our conventionally-driven ladder, but it has now picked up two zeroes.

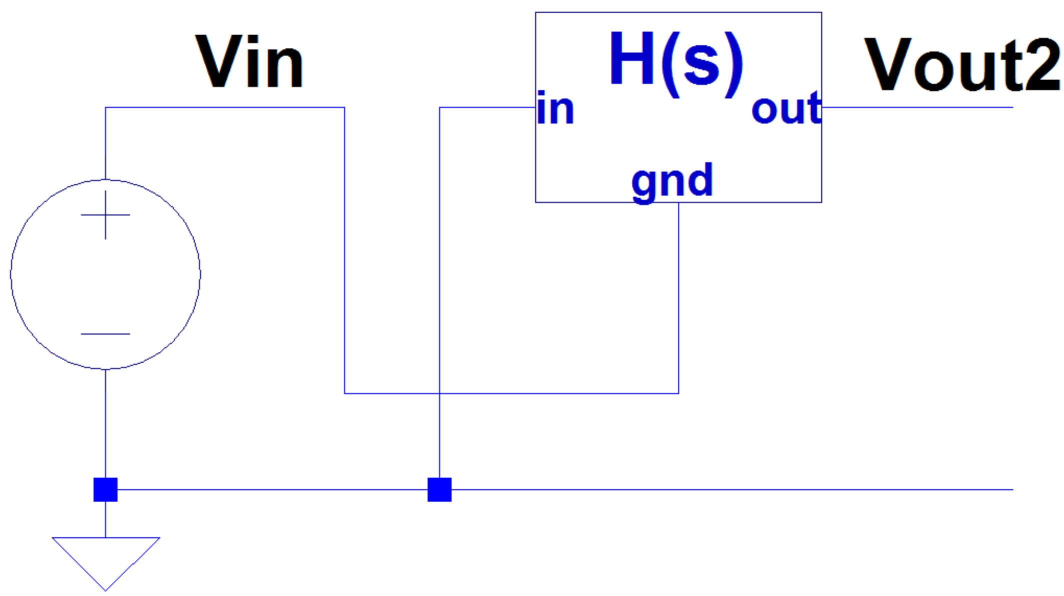

Figure 6: The general case of the connection shown in figure 5.

A very easy bit of circuit analysis enables us to figure out what these zeroes are. We can do this for the general case shown in figure 6, for an arbitrary transfer function H(s), to get a simple expression for the effective transfer function to voltage V_{out2}:

$$V_{out2} - V_{in} = H(s)\cdot(0 - V_{in})$$
$$\therefore V_{out2} = V_{in} - H(s)\cdot V_{in}$$
$$\frac{V_{out2}}{V_{in}} = 1 - H(s) \qquad [1]$$

If we let our transfer function H(s) be a normalized second-order lowpass with a Q of one-third, as we get from the unit-component case, we can write down

$$H(s) = \frac{1}{s^2 + 3s + 1}$$
$$\therefore\ 1 - H(s) = \frac{(s^2 + 3s + 1) - 1}{s^2 + 3s + 1} = \frac{s(s+3)}{s^2 + 3s + 1} \qquad [2]$$

So there's a zero at DC, and another one at three times the frequency given by the RC product of the network, which is unity in our simple case. Over a narrow band of frequencies, these zeroes push the gain up by more than the poles are pulling it down, and the result is actual gain. If we were to multiply the second resistor by a factor of say one million, and divide the second capacitor by that same factor, we'd get very near to a Q-factor of one-half, and we'd get a peak gain of 1.25 dB. This is the best we can do with four passive components. All circuits built this way, starting from a lowpass RC ladder, have a highpass response that falls towards DC at a mere 6 dB per octave. However many poles are 'on the bottom', there's always just one fewer zero 'on the top'.

What's the highest possible peak gain value that we could get? That can be answered without any algebra. If our function H(s) is any transfer function that has no gain peak in it, then its maximum magnitude is unity. What's the worst phase shift it can have? Well, a signal can never be any more out of phase than 180 degrees, which is equivalent to an inversion. This makes the maximum possible magnitude of 1-H(s) equal to 2. So such a circuit can never give you a peak of more than +6 dB, however you build H(s).

Using a ladder containing four resistors and four capacitors, it can be shown (though I'm not going to) that we can get to a maximum peak of 2.8 dB. But if we're going to allow the use of four resistors and four capacitors, we can do much better by connecting them up differently. The circuit we've looked at here loses the low frequency portion of our signal. We might want that DC value for some reason. Remember the extra credit question? Well, it turns out that we can construct another circuit, related to this one, which never has less than unity gain at any finite frequency, and can have a significant gain peak over a working range of frequencies.

Examine figure 7, which shows the general case for this method. Instead of using a single H(s) block (typically our RC ladder), we can make up two identical blocks and connect them in antiparallel at their input ports. One of them is connected conventionally, as was done in figure 1, and the other is connected up with the input flipped, as was done in figure 5. Then, we look at the output voltage difference between the end nodes of each block. This connection is a generalization of a so-called lattice network.

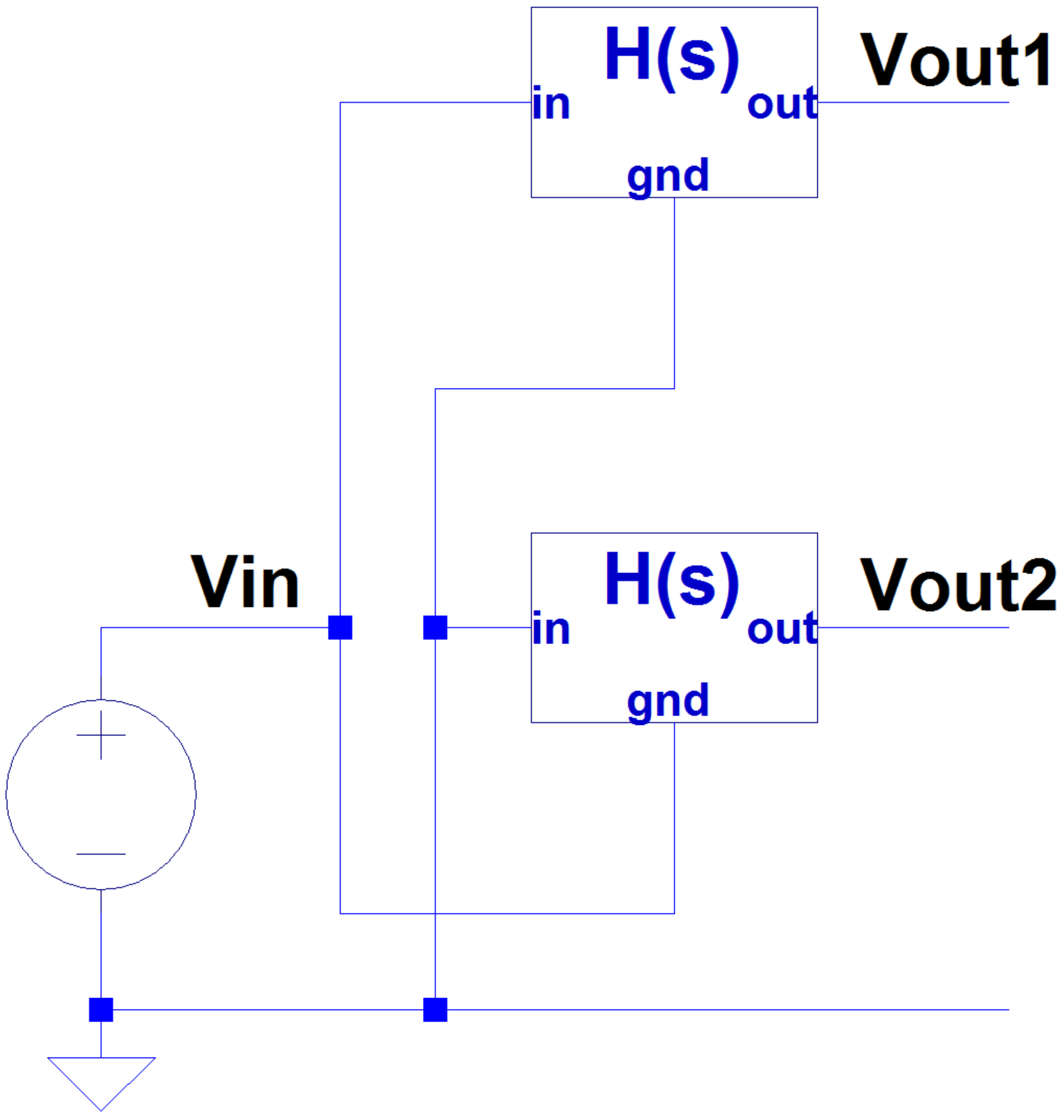

Figure 7: The general case for the lattice method of connecting two blocks.

The frequency response between V_{out1} and V_{out2}, when we use our figure 1 equal-value RCRC ladder to make H(s), is shown in figure 8 (along with the two separate node voltages), and it's very different from that in figure 4. Instead of being a highpass response, with no signal coming through at DC, it's now a 'bump' response. It has a response peak that has 1.6 dB gain, using these values, with unity gain both at DC and at very high frequencies. This circuit has some voltage gain at every possible finite

frequency (though not a lot when you're a long way from the peak value, clearly). Not bad for a circuit that's built with just resistors and capacitors, eh?

The gain at the response peak, expressed in dB, is over twice the peak value given by the single back-driven ladder of figure 3. By scaling the second half of each ladder network to get close to the limit Q of one-half, we can get to a peak gain of 3 dB. The absolute limit, for an arbitrary H(s) that never itself gets above unity gain, is now a linear gain of 3, or 9.54 dB in logarithmic money. This 'falls out' of the analysis for the transfer function of this circuit, which is easy to derive. The transfer function from the input to V_{out1} is H(s), obviously, and we already showed that it's 1-H(s) to V_{out2}. So the transfer function to the difference of the voltages at these two nodes is trivially 2H(s)-1, and the maximum and minimum magnitudes of this function are 3 and 1, given the no-peaking constraint on H(s). At very low frequencies the output voltage is in phase with the input (|H(s)| tends to unity as frequency tends to zero), while at very high frequencies it's out of phase (|H(s)| tends to zero as frequency tends to infinity). This implies that the resultant function is not minimum phase, and that's a general characteristic of lattice networks, which are often used to create passive allpass filters.

It's interesting to note that this is real signal gain, in voltage terms. For an arbitrary input signal, the magnitude of every possible Fourier component of an input signal is increased in value, never reduced. This circuit (with the values shown) unconditionally increases the integrated spectral density of an input voltage signal.

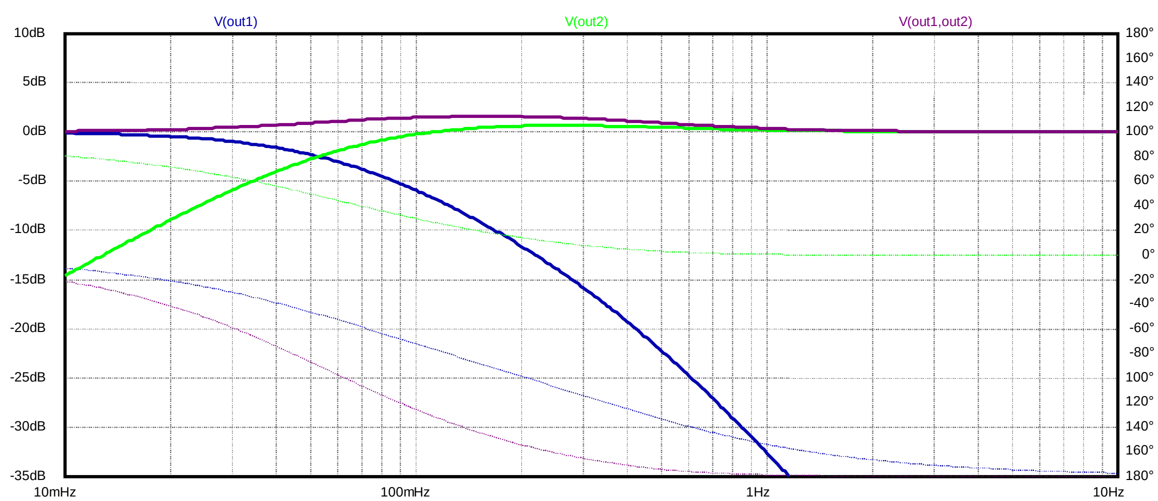

Figure 8: The response (purple) for figure 7 when H(s) is our unit component ladder.

For the particular case of a second-order lowpass H(s), the resultant transfer function has some interesting properties. A bit of algebration and a whiff of calculus can be used to show that the peak in the response always occurs at the pole frequency of the H(s) you use. The linear magnitude of the gain at the peak is equal to sqrt($1+4Q^2$). In the equal-component case, this gives a value of 1.202, which fortunately matches the 1.6 dB that we see from the simulation. We can push the gain up to an impressive 3 dB by suitably scaling up the impedance level of the second half of the RC ladder.

So, there we are. Four resistors and four capacitors used; a peak AC voltage gain of up to 3 dB, with unity gain to both DC and very high frequencies (which come out inverted). Well, it's not Earth-shattering, I know. But I bet there's a lot of you out there that never knew that you could get any unconditional voltage gain at all out of just a bunch of Rs and Cs. So I don't feel too bad about it as a Filter Wizard topic!

If you're a glutton for punishment, figures 9 and 10 show the circuit and response of a version using six-pole ladder networks, giving a gain peak of 6dB.

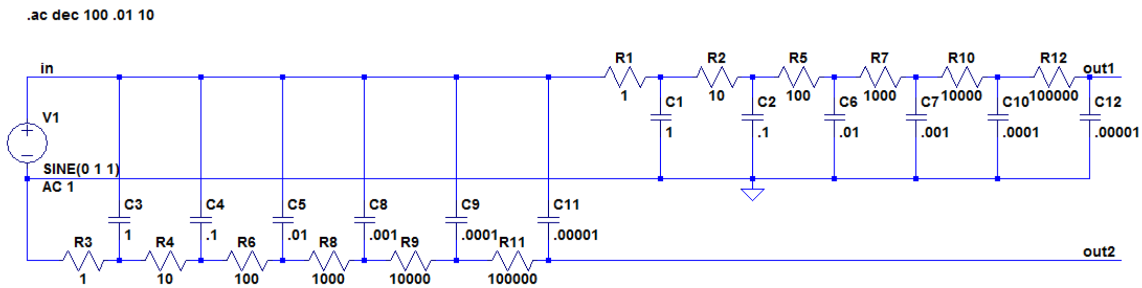

Figure 9: Using two six-pole ladder networks.

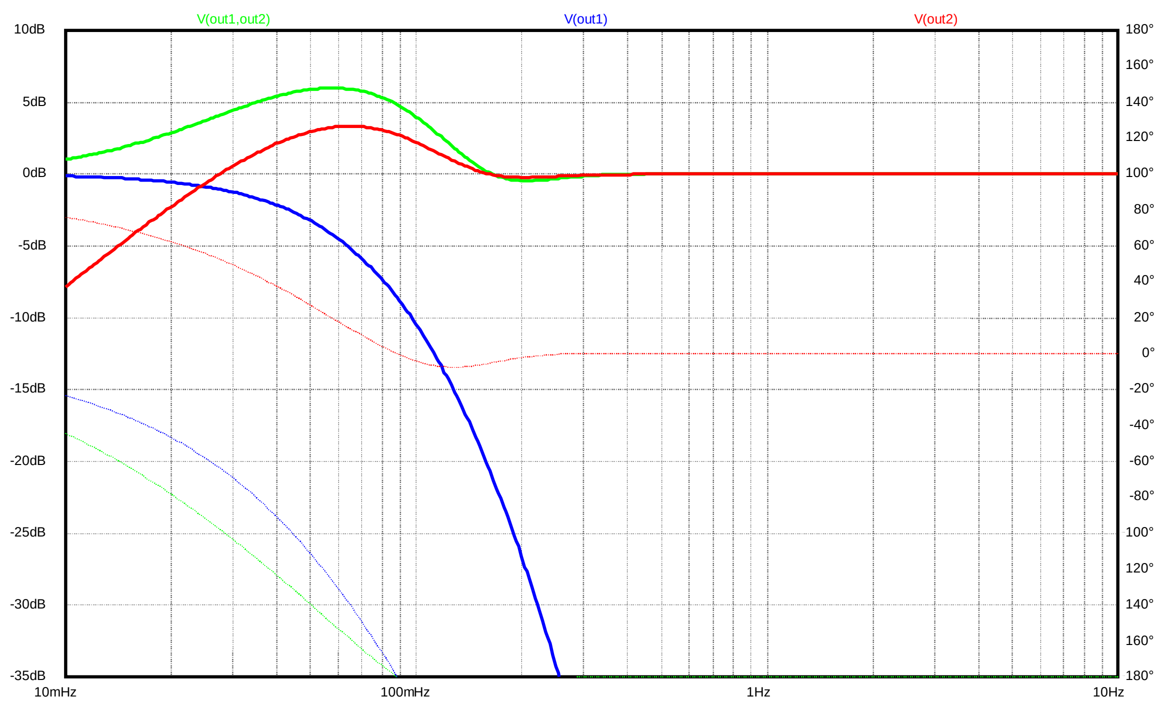

Figure 10: response of the circuit in figure 9 – green trace is V_{out1}-V_{out2}.

Is this circuit useful for anything? Not as far as I've been yet able to determine (submissions welcomed, of course). Circuits akin to this can be found at the core of some RC oscillator topologies, if you're prepared to look sideways, backwards or in some other unconventional way at the networks of passive components within them. One unfortunate thing is that the circuit can't deliver power gain, unlike LC networks, which don't absorb any of the energy flowing through and around them. The behaviour of the network goes awry when you start to introduce finite source and load impedances. I'll

leave you to do some playing around with that, if you're one of the small cohort of readers who are all fired up about this and might be able to jazz up their next project with it. For the rest of you, lattice hope that it comes up in your next job interview! best / Kendall